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Statistical Properties of Strings

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Abstract

We investigate numerically the configurational statistics of strings. The algorithm models an ensemble of global $U(1)$ cosmic strings, or equivalently vortices in superfluid ^4He . We use a new method which avoids the specification of boundary conditions on the lattice. We therefore do not have the artificial distinction between short and long string loops or a ‘second phase’ in the string network statistics associated with strings winding around a toroidal lattice. Our lattice is also tetrahedral, which avoids ambiguities associated with the cubic lattices of previous work. We find that the percentage of infinite string is somewhat lower than on cubic lattices, 63% instead of 80%. We also investigate the Hagedorn transition, at which infinite strings percolate, controlling the string density by rendering one of the equilibrium states more probable. We measure the percolation threshold, the

critical exponent associated with the divergence of a suitably defined susceptibility of the string loops, and that associated with the divergence of the correlation length.

Line defects are formed after a phase transition if the manifold of equilibrium states M (the vacuum manifold) is not simply connected [1]. They are studied theoretically in the context of field theories in the early Universe under the name of cosmic strings, but also exist in the laboratory in the form of superfluid vortices, superconductor flux tubes, dislocations, and line disclinations in nematic liquid crystals. The interest in the formation of defects in cosmological phase transitions has been reflected in laboratory experiments studying the formation and evolution of defects in nematic liquid crystals [2, 3] and He-II [4].

In this paper we look at the configurational statistical of $U(1)$ strings, where M is a circle. The properties of ensembles of 1-dimensional objects have importance in many problems in physics: polymer science [5]; dislocation melting [6]; the liquid-gas transition [7]; and the Hagedorn transitions in effective [8] and fundamental [9, 10] theories of strings at high temperature. The notion that the entropy associated with the strings plays a key role in the superfluid phase transition goes back to Feynman and Cohen [11], and has been expressed in a field theory context by Copeland et al. [12]. Strings are usually modelled by random walks, either Brownian or self-avoiding. A self-avoiding walk (SAW) models an excluded volume effect, and is known to apply well to polymers [5]. However, it is not clear that either kind of walk represents the configurational statistics of cosmic strings or superfluid vortices, for there are long-range interactions which could change the Hausdorff dimension. In fact, we find that at low density strings are *self-seeking*: that is, their fractal dimension is higher than 2.

The formation of $U(1)$ strings has been studied by several workers in simulations on cubic lattices [14, 15, 16]. Kibble [17] has studied $SU(2)$ strings, and others have studied composite defects such as monopoles joined by strings [18]. Much of this work has been done on finite size lattices with periodic boundary conditions, which suffers from boundary effects that have nothing to do with the true ensemble [16, 19]. We propose a formalism which can simulate an arbitrarily large lattice, without considering more points than necessary to follow a single string. This allows us to follow string statistics to string lengths of 10^5 lattice units, without prohibitive memory requirements. This is a necessity for the accurate determination of the statistical measures of the model: the fraction of the total string mass contained in infinite strings; the percolation threshold at which this fraction vanishes; the exponent for divergence of the susceptibility at this threshold; and the Hausdorff dimension of the strings.

Before we describe the numeric procedure in detail, we will motivate our measurements in the following section and explain what results one expects intuitively.

1 String Statistics and the Scaling Hypothesis

One might expect the network of cosmic strings to have the statistical properties of a self-avoiding random walk. A SAW builds up an excluded volume as it follows its path, which is, in a statistical sense, spherically symmetric and clustered around the origin. The SAW therefore has a stronger tendency to move away from the origin than the Brownian walk, which is allowed to intersect itself arbitrarily often. This property is expressed in the fractal dimension D of the walk, which is the exponent relating the average string length l between two points on the same string to their relative distance R by

$$l \propto R^D . \quad (1)$$

It is well known that the dimension for a Brownian walk is $D = 2$ and for a self-avoiding random walk $D = 1/\nu = 1/(0.5879 \pm 0.0005)$ (see ref. [13] and references therein for a summary of different methods used to obtain that result). However, the original string formation simulations [14, 15] are consistent with $D = 2$: this is because they simulated a dense string network. A single string, as we trace out its path, experiences a repulsion from all of the segments of other strings, which do not have any statistical bias towards the origin. Therefore the repulsion from the forbidden volume will also have no directional bias. Thus the fractal dimension of the string is expected to be two. In polymer physics, this effect has been known for some time to occur in dense solution of polymers, while a polymer in a dilute solution exhibits the statistics of a self-avoiding random walk [5]. In a statistical sense, the network of cosmic strings was argued to be equivalent to a dense network of polymers [21].

We can introduce the scaling hypothesis in order to estimate a few other properties of the string network. Scale invariance means that the string network looks the same on all scales in terms of statistical properties. In fact Brownian random walks are scale invariant. With this hypothesis, the expected distribution of closed loops can be derived [22]. From dimensional arguments, the number of closed loops with size from R to $R+dR$ per unit volume can be written as

$$dn = \frac{dR}{R^4} f\left(\frac{R}{\xi}\right) .$$

If the system is scale invariant, the distribution should be independent of the correlation length ξ , and one expects

$$dn \propto R^{-4} dR . \quad (2)$$

The length distribution of loops for strings with a fractal dimension of D is therefore

$$dn \propto l^{-b} dl, \quad (3)$$

with $b = 1 + 3/D$. It was originally expected [14] that from scale invariance there should be no infinite strings. This turned out not to be the case, since infinite strings can still look statistically the same on all scales: a Brownian walk is scale invariant and has a non-zero probability not to return to the origin in $d > 2$ dimensions. The origin of the scale invariance of the string network seems to be connected with the absence of long-range correlations in the order parameter [14]. However, scale invariance does not necessarily imply that the network is Brownian as originally stated: it can be seen above that one does not need $D = 2$ in order to have a scale-free distribution of loop sizes R .

Vachaspati [15] devised an algorithm that induced correlations in the order parameter by lifting the degeneracy of the manifold of equilibrium states. This had the effect of reducing the density of string, but *increasing* the dimension D , which argues against the identification of strings with SAWs. He found that there was a critical density below which there were no “infinite” strings, which for a finite size lattice of dimensions N^3 are all strings longer than kN^2 , with $k = O(1)$ ¹. In the low density phase there is a scale c which appears in the loop length distribution,

$$dn = al^{-b}e^{-cl}dl, \quad (4)$$

as a cut-off. As the critical density is approached from below, $c \rightarrow 0$, and the mean square fluctuation in the loop length

$$S = \langle l^2 \rangle - \langle l \rangle^2, \quad (5)$$

diverges (see Figure 7). This divergence signals a phase transition, called the Hagedorn transition, which arises from a density of states which is exponentially increasing with energy. As mentioned in the introduction, it has been implicated in many branches of physics, although Vachaspati’s algorithm most directly models the symmetry-breaking phase transition in $U(1)$ scalar field theory or the normal-superfluid transition in liquid helium. Previous studies [12] assumed from the outset that the strings are Brownian, and tried to model the true string ensemble by adding in interactions. Vachaspati’s algorithm enables us to measure directly the string statistics such as the critical density, the dimension, and the critical exponents, and to test the validity of the hypothesis of scale invariance.

¹The measured percentage of mass in infinite strings is rather insensitive to the particular choice of k [15]. However, string loops which wind around the toroidal lattice create difficulties in the interpretation of the variables in Eq. (1). This explains why our measurements disagree with [16], who considered winding loops to be a new long-string phase with different scaling exponents.

2 Choosing a lattice

The general idea of such simulations is to put a grid onto space, with the average nearest neighbour distance representing the correlation length of the order parameter. Now, to every vertex of the lattice, we can assign a random value to the order parameter (here a complex scalar field), subject to the constraint that it lie in the manifold of minima of the free energy M . The correlation length ξ is certainly less than or equal to the cosmological horizon distance ct at the time of the phase transition. In fact, in a second order transition, it is just the Compton wavelength of the scalar particle, while in a first order transition, such as occurs in a liquid crystal, it is interpreted as the average bubble separation. Thus in the second order case we are simulating the thermal ensemble, and in the first order case we are simulating the nucleation of bubbles with random phases. Furthermore, following Vachaspati and Vilenkin [14], we discretize M by allowing the phases of the scalar field at the vertices to take only one of the three values: $\theta = 0, 2\pi/3, 4\pi/3$. A string passes through the face of the space-grid, if the scalar field rotates through 2π when traced around the edges of that face. In the process of tracing the field between two vertices of the lattice, we assume that the interpolation between the two values assigned to the vertices is such that the change of θ is minimal – the so-called geodesic rule [1, 23, 24]. If we apply this rule, a face is penetrated exactly if all the three possible values of θ occur on the face in question. It is easy to convince oneself that for a tetrahedron this has to be the case for either two or none of the four faces, thus the geodesic rule ensures conservation of the ‘string flux’ and no string can be terminated except by building a closed loop. Discretizing M is numerically sensible, for it allows the use of integer arithmetic, but it does change the probability that a string passes through a face [25]. It may also change the percolation probability, and possibly even the critical exponents. This is something we plan to investigate in future.

Using this construction to simulate string formation on a cubic lattice leaves ambiguities in the definition of the network, since some unit cells will be penetrated by two strings. In this case we would have to assign connections of the two incoming and two outgoing segments randomly. It is not obvious *a priori* with what probabilities to make these assignments. Vachaspati and Vilenkin chose equal probabilities, and found a long string density significantly higher than ours (see below). It seems likely that connecting strings in adjacent faces more often will make loops more common.

A tetrahedral lattice does not suffer from this problem. One has to be quite careful to choose a tetrahedral lattice which produces rotationally invariant string statistics. No tetrahedral lattice which is only a subdivision of a cubic lattice can achieve that. We made measurements with a cubic lattice split up into the same set of six tetrahedra, as well as

with a lattice on which the subdivision of the cubic cell into tetrahedra was programmed to happen randomly. In either case one body diagonal is always preferred, for it occurs either in all of the cubes or in none of them. Isotropy of the string distribution was measured by an inertia tensor. Instead of measuring only R^2 , as is necessary to measure the fractal dimension and test the validity of Eq. (1), we measured $I_{ij}(l) = \langle R_i(l)R_j(l) \rangle$. In the case of any subdivided cubic lattice this tensor turned out not to be proportional to the unit matrix δ_{ij} , as one must have for a statistically isotropic string distribution.

Among the tetrahedral lattices we tested, only the one which is the dual lattice to a tetrakaidekahedral lattice proved to have this property.² This is in effect a tetrahedral lattice with a set of vertices building up a body-centered cubic lattice. The unit cell of the tetrahedral lattice is shown in Figure 1 with the twelve equivalent tetrahedra it consists of. Another useful feature of this lattice is [21] that the edges of all the tetrahedra are nearly equally long and therefore can be said to represent a given correlation length ξ rather well. There are only two different edge lengths with a length ratio $2 : \sqrt{3}$. This is also reflected in the fact that the first Brillouin zone of the body-centered cubic lattice, which builds up the tetrakaidekahedral lattice, is nearly spherical. The dual lattice to the tetrahedral lattice is shown in Figure 2. The possible string paths lie on the edges of the dual lattice, while the field values are assigned to the center of each tetrakaidekahedron.

In our measurements, we started out by considering a single tetrahedron being penetrated by a string. We then followed the string along whichever path it was forced to take by the random field value assigned to the new vertex of the next tetrahedron penetrated. At every step we have to check whether we have been at the presently relevant space-point already. Since 24 tetrahedra are adjacent to each lattice point on the body-centered cubic lattice, the field value at a single lattice point can be required several times while tracing a string. To be able to recall these values without wasting memory on lattice points which are never needed, we created a linked list of structures, whose elements carried the essential information of the already used lattice points. The advantage lies in the realization that lattice points which are not adjacent to the string under consideration are irrelevant, and that the way the string is constructed itself ensures that the string “knows” about the string density around it. This is the crucial improvement in our treatment. The computational effort to check the whole list of structures at every step increases proportionally to the square of the maximum length of the strings. We created a linked list of such linked lists, and searched in the list containing only certain elements,

²A random lattice with a proper triangulation would also satisfy the requirement of statistical isotropy, but such a simulation would be much more computationally intensive. Also, the lattice spacings would differ to a larger extent, which would represent a first order phase transition to produce the strings after (randomly distributed) bubble collisions. We don’t expect any different results on a random lattice.

e.g. only elements within a certain distance interval from the origin. This is effectively the same as using a very undersized hash table with collision resolution by chaining [26]. While this work is being completed, the authors plan for future work to use an oversized array as a hash table, allowing for a “nearly unique” hash function which, compared to the simple distance-mapping we used here, produces many fewer data collisions and will reduce the search time to the order of the time it takes to write down the new step, so that the computational effort of tracing a string will be directly proportional to the string length. However, even our undersized hash table speeded up the computation by a factor of about 700 compared to simple chaining and were able to trace strings up to lengths of 100,000 correlation lengths, and to do statistics over 10,000 of these strings, in about 24 hours on a HP 715/50. In fact, with our algorithm the length of traceable strings is limited only by the memory required to store *one* string. In comparison with previous calculations this allows very good statistics on long strings without running out of memory. If one creates a finite size lattice and assigns field values to all the points on the lattice before tracing out the strings (as it was done in all the previous work on this subject), the longest strings occurring (regardless of the boundary conditions) are expected to be about N^2 correlation lengths long, if N^3 is the number of lattice points. In order to achieve the same maximum length, one would therefore have to use a 300^3 lattice. If one counts strings near N^2 correlation lengths on a N^3 lattice, one either gets spurious effects from the boundary conditions [16], or one faces problems of biased statistics.³

3 Results for Equiprobable Field Values

We first set about measuring the statistics for equiprobable field values, that is, that all three values of the field had the same probability at each vertex. This models the statistics of the string network at formation. In Figure 3 we plotted R^2 against l , which should be exactly linear in the Brownian case (and therefore it should have slope one on the logarithmic plot). In fact we measure for the fractal dimension of the $U(1)$ strings

$$D = 2.01 \pm 0.01 , \tag{6}$$

which indicates no repulsion from the origin, but perfect Brownian statistics instead. We detect no deviations from a perfect scaling behaviour, and as soon as the length of

³ For example, with absorbing boundaries we count only the very crumpled strings among the very long ones, and the measured Hausdorff dimension of the long strings will therefore be larger than the actual one

string under consideration is an order of magnitude longer than the lattice spacing, this coefficient is completely independent of the length of the strings considered.

There is a subtlety to discuss: even though the scaling might be Brownian, this equation cannot be understood as “after l steps one is likely to be at the distance R from the origin”, since there is a certain probability that the walk has terminated by then, which does not happen to a Brownian walk. We obviously only average over the strings that have survived up to the distance l . Of course, for Brownian walks, if we average them only over the ensemble that has not revisited the origin until it got to a length l we still get a fractal dimension $D = 2$, because D is measured in the asymptotically large l -limit, where the probability of return to the origin tends to zero *for all* Brownian walks, whether they have revisited the origin in the past or not. A Brownian walk is Markovian and does not remember its past, so every walk will revisit the origin only a finite number of times, if the fraction of walks that never come back is nonzero, which is trivially true. The probability of return to the origin, however, depends very much on the microscopic details of the lattice, for instance on whether we allow “full turns” or not. For example, as a test we produced our random walks the same way as we produced the cosmic strings, but did not store the phases if we revisited a given lattice site. This disallows full turns, but does not force the walk to avoid itself. The total walk mass in loops was then only $\approx 9\%$. Conversely, since the coordination number of the lattice on which our strings live is four, a quarter of all the walk mass would be in loops of only the smallest possible size (length 2), if we allow immediate returns on the random walk.

We measure that $(63.3 \pm 1.0)\%$ of the total string mass is in infinite strings, compared to the Vachaspati and Vilenkin result of about 80% [14]. This is an indication that the random assignment of string elements in cubes that are crossed by more than one string is statistically misleading, and that strings prefer to crumple up to a larger extent than is assumed by such a random assignment. Unfortunately this is not a quantitatively hard result (and neither was the one of [14]), for reasons that are mentioned above. Some of the difference in our results may come therefore from the lower coordination number of our dual lattice, so that short loops are more probable than on a cubic lattice (4/125 of all the string mass is in four-step loops on a cubic lattice, whereas on our lattice this ratio is 1/27). Another reason why the mass in small loops affects our results is that most of the string mass that is in loops, is in small loops, and our lattice allows loop diameters that are $1/\sqrt{3}$ times smaller (in units of correlation lengths) than the smallest loops allowed in ref. [14]. In our formalism it is easy to convince oneself that these 63.3% are simply equal to the ratio of strings which have not terminated and run into themselves after the maximum number of steps. It should be mentioned that at string lengths of 100,000 ξ this is very insensitive to the cutoff-length of our calculation. The discretization of

the symmetry group and of 3-space is certainly introducing bigger inaccuracies than the cutoff at that length.

To measure the fraction of loops within a certain length-interval, i.e. the exponent in Eq. (3), we use the linear fit in Figure 4, which is a logarithmic plot of the density of string elements in loops of a certain length interval. From (3) this relation should be

$$dN \propto l^{1-b} dl , \quad (7)$$

since the number of loops of a certain length is $dn = dN/l$. Both the scaling hypothesis and the assumption of Brownian statistics seem to be good working hypotheses for $U(1)$ strings at formation.

4 Low String Densities: the Hagedorn Transition

Drawing on lessons learned from polymer statistics, the fact that our algorithm generates Brownian strings is a result of the dense packing of the strings. Naturally the question of what happens for low string densities arises: we would like to find the “natural” dimension of a stringy random walk. There is also an important phase transition to investigate, that between a percolating phase containing a certain fraction of infinite string, and a low string density phase consisting of finite loops only. In string theory this is known as the Hagedorn transition [8, 9, 10]. It is also of importance in spontaneously broken field theories with vortex solutions [12], for the stringy degrees of freedom dominate the fluctuations near the phase transition, and generate a divergence in the susceptibility slightly below the mean-field critical temperature. The question also arises of the order of the transition. The calculations start from the assumption that the string ensemble is Brownian, and then include the effect of interactions between the strings in an approximate way. They find evidence that string-string interactions turn the second-order transition obtained in the mean-field approach into a first-order one.

Vachaspati [15] proposed an algorithm for reducing the string density, which selects one of the three allowed field values, say 0, as being more probable. This reduces the probability of a string penetrating the face of a lattice. Thus we can generate an ensemble with the *average* string density fixed at will. (Physically one can think of this as applying an external field, which spoils the $U(1)$ symmetry of the state.) When the probability of string formation is made sufficiently small, the strings stop percolating. One also finds that the strings are more crumpled, that is, they have dimension higher than 2. Thus strings are not at all like polymers – instead of being self-avoiding they are ‘self-seeking’ at low densities. It is easy to convince oneself that this is true by virtue of our lattice construction, and that this property must increase with increasing η .

We introduce a bias parameter η and set the probabilities $p(k)$ of the vacuum phase $2k\pi/3$ as follows: $p(0) = \eta/3$, $p(1) = p(2) = \frac{1}{2}(1 - \eta/3)$. The probability P_f for a string to pass through a face is therefore $6p(0)p(1)p(2) = \eta(1 - \eta/3)^2/2$. The probability P_c for a string to pass through a cell is $2P_f$, as the following argument makes clear. If a string pass through a face, it must emerge through one of the other faces of a tetrahedral cell with probability $1/3$. Thus the probability that a string passes through a particular pair of faces is $P_f/3$. Since there are 6 pairs of faces, the total probability is $2P_f$. Therefore the string line density is

$$\rho = \eta(1 - \eta/3)^2 \quad (8)$$

in lattice units. We can also express the expectation value of the order parameter as a function of η :

$$\langle \phi \rangle = (\eta - 1)/2. \quad (9)$$

Thus we see that, averaged over many lattice sites, the symmetry is unbroken at $\eta = 1$, moving towards $\langle \phi \rangle = 1$ as $\eta \rightarrow 3$.

For low η , the string network maintains a fraction of infinite string, and the equations (1) and (3) hold with η dependent scaling exponents. The equation which encodes the scale-free hypothesis for the loop distribution, Eq. (2), is not very well satisfied: the scaling relation $b = 1 + 3/D$ is satisfied with errors of about 5%, while the estimated statistical error is just over 1%. It is, of course, possible that we have underestimated our errors. We record our measurements in the Table 1.

η	$N(l > 25000)$ in %	D	ρ	b	$1 + 3/D$
1.0	63.1 ± 1.0	2.002 ± 0.004	0.4444	2.49 ± 0.02	2.50
1.1	59.5 ± 1.8	2.03 ± 0.03	0.4412	2.47 ± 0.03	2.48
1.2	44 ± 1	2.19 ± 0.03	0.4320	2.32 ± 0.03	2.37
1.21	41 ± 1	2.23 ± 0.02	0.4308	2.22 ± 0.02	2.35
1.22	38 ± 1	2.28 ± 0.03	0.4295	2.20 ± 0.02	2.32
1.23	33 ± 1	2.31 ± 0.03	0.4282	2.18 ± 0.02	2.30
1.24	28 ± 1	2.38 ± 0.03	0.4268	2.14 ± 0.02	2.26
1.25	25 ± 1	2.42 ± 0.03	0.4253	2.11 ± 0.02	2.24
1.26	19 ± 1	2.53 ± 0.03	0.4238	2.09 ± 0.02	2.19
1.27	14 ± 1	2.66 ± 0.03	0.4223	2.05 ± 0.02	2.13
1.28	10 ± 1	2.71 ± 0.03	0.4208	2.04 ± 0.02	2.11

Table 1 : Scaling exponents and the string density (in units of string elements per tetrahedron) for different values of $\eta < \eta_c$.

These values up to $\eta \approx 1.25$ seem to all represent one “high temperature” phase of the network, and the statistics do not change qualitatively until we reach the percolation threshold, where there is no longer any infinite string.

In that domain Eq. (3) has to be replaced by Eq. (4), and scale invariance is no longer satisfied. Instead, the parameter c is the inverse of a diverging length scale, as it appears in any second order phase transition. Besides the exponents in Eq. (4) one can measure the susceptibility of the string network as defined in Eq. (5) and the average length of string loops. We list the measurements in the Table 2. We did observe evidence for non-analyticity in the average string length at the critical η , which may not be obvious from the table below. Better statistics are needed to determine the (small and positive) critical exponent for $\langle l \rangle$.

η	$\langle l \rangle$	$\langle l^2 \rangle$	S	b	c
2.2	10.4404	154.04	45.0386	1.635	0.0233
2.1	11.649	207.831	72.1329	1.772	0.0157
2.0	12.7826	273.722	110.329	1.687	0.0122
1.9	14.3919	385.025	177.898	1.694	0.0089
1.8	16.5119	584.274	311.633	1.776	0.00541
1.7	18.8471	940.33	585.117	1.802	0.00286
1.6	21.4212	1617.23	1158.36	1.838	0.00144
1.55	23.3261	2358.77	1814.67	1.958	0.00104
1.5	25.4684	3567.08	2918.44	1.989	0.00064
1.45	27.932	5970.12	5189.9	1.977	0.00038
1.4	30.5805	11442	10506.8	1.969	0.000193
1.38	30.7761	14757.6	13810.4	1.998	0.000129
1.37	32.1548	18380.6	17346.6	1.986	0.000111
1.36	33.84	23280.3	22135.1	1.979	0.000091
1.35	34.1428	28543	27377.2	2.004	0.000067
1.34	34.0535	36636	35476.4	2.008	0.000048
1.33	34.8022	47241.1	46029.9	2.025	0.000035
1.32	34.9562	60152.3	58930.3	2.042	0.000024
1.31	35.1275	79784.5	78550.5	2.045	0.000016
1.30	35.9985	120865	119569	2.070	0.000009
1.29	37.1514	200032	1986528	2.080	0.000005
1.28	36.8624	348316	346957	2.085	0.000002

Table 2 : The relevant measurements for $\eta > \eta_c$. The divergence of the length scale $1/c$ as well as the

divergence of the susceptibility indicate a second order phase transition at η_c .

The assumption that the divergence of the susceptibility has a critical exponent γ , so that

$$S = S_0(\eta - \eta_c)^{-\gamma}, \quad (10)$$

and that similar exponents exist for the other variables in Table 2 gives a good estimate of the critical η . We measure

$$\eta_c = 1.243, \quad \gamma = 2.36.$$

There is also an exponent σ for the loop size cut-off parameter c , defined by

$$c \propto |\eta - \eta_c|^{\frac{1}{\sigma}}. \quad (11)$$

The divergence of $1/c$ can be seen in Figure 7. A fit to this form for c gives

$$\eta_c = 1.256 \quad \sigma = 0.397. \quad (12)$$

Thus we estimate $\eta_c = 1.25 \pm 0.01$. The errors in the exponents are more difficult to gauge, for they depend on the uncertainty in η , and we have not performed a proper error analysis. However, we believe that they can be trusted to about 5%.

5 Comparison to Thermal Quantities

If we define a probability density that a given string loop is in the length interval $[l, l + dl]$ by $P(l) \propto dn/dl$ and the normalization condition $\sum_l P(l) = 1$, we can define an information entropy, contained in the probability distribution function by

$$S = - \sum_l P(l) \ln P(l) .$$

The percolation transition that we observe is not a thermal phase transition, but we can attempt to relate certain observables in our Monte Carlo ensemble to variables in a canonical thermal ensemble. We can for instance try to relate the average length of string loops to an average energy of a thermal ensemble (our susceptibility in Eq. (5) relates then to a specific heat, which is indeed the case for perfect $U(1)$ strings). It would then make sense to define a temperature θ as

$$\frac{1}{\theta} = \frac{\partial S}{\partial \langle l \rangle} ,$$

and see how the system behaves under changes of this parameter. Unfortunately we do not yet have sufficient statistics, since quantities like $\langle l \rangle$ in Table 2 are not even monotonically decreasing with increasing η . The necessity of improved statistics is also apparent in Table 3, which lists the information entropy of our Monte Carlo ensembles at different values of η . The finite difference approximation to $[\partial S / \partial \langle l \rangle]^{-1}$ is statistically quite unstable because both S and $\langle l \rangle$ have small differentials compared to their absolute values. Improved statistics will be obtainable with the algorithm under development, which will use a bigger hashtable, with less hash-function collisions.

η	S	$\langle l \rangle$
1.28	1.19223	36.8624
1.29	1.18510	37.1514
1.3	1.16673	35.9985
1.31	1.15198	35.1275
1.32	1.14534	34.9562
1.33	1.14681	34.8022
1.34	1.14117	34.0535
1.35	1.14373	34.1428
1.36	1.14637	33.8400
1.37	1.14292	32.1548
1.38	1.14067	30.7761

Table 3 : The values of the entropy and the average loop length in our Monte Carlo ensembles. They are rather well defined, although the ratio of their differentials – corresponding to a thermodynamic temperature – has still very bad statistical errors.

6 Conclusion

We can apply our numerical technique to other vacuum manifolds, whose non-contractible contours have a discrete mapping onto triangular faces, such as the vacuum manifold of nematic liquid crystals (NLCs), which is RP^2 , or $SO(3)/O(2)$, the space points on a two-sphere with opposite points identified (it is therefore also equivalent to S^2/Z_2). The statistics there could be directly verified in the laboratory, and some variables, like the average number of strings created per bubble after a first order phase transition into the nematic phase, have in fact been measured already [3]. The symmetry group of NLCs is very easily understood in terms of the orientations of the molecules. The symmetry

manifold consists of the orientations of a directionless rod. Many NLCs are polarisable, so applying a strong electric field could make the molecules prefer orientations parallel to the field axis. Thus one could measure things like the percolation threshold and the exponents such as D and b . Work in this direction is in progress.

We think that the numeric measurement of configurational statistics has never been done with such accuracy, since the numerical methods used here allow a much higher number of very large strings to be investigated, without being influenced by boundary conditions. This gets rid of the adverse effects of the topology of the lattice, and also speeds up the computations considerably. We are able to confirm the scaling hypothesis for the case of an exact $U(1)$ symmetry, and we find no compelling evidence that the fractal dimension of the cosmic string is different from 2.

As the string density decreases, we measure an increase in the fractal dimension, which implies that the strings get more crumpled and are statistically biased to turn back onto themselves. This was indeed implicit in Vachaspati's results [15]. Interestingly this effect has also been shown to occur in the equilibrated phase of flat-spacetime string dynamics. In ref. [27] it is shown that for a sufficiently high initial string density the string dynamics keep a certain fraction of string mass alive in infinite strings, for reasons that lie beyond simple flux-conservation arguments. Furthermore, ref. [28] confirms that the Equations (3) and (4) hold for the high and low density phase of string dynamics respectively, but with a constant exponent of $b = \frac{5}{2}$ in both phases, as one would expect for a thermalized ensemble [10]. For a thermalized ensemble of string one also expects the density in string loops to be constant for all string densities above the critical density, as numerically confirmed in ref. [28]. That this is not the case for the cosmological initial conditions has been postulated in [15] and had received further backing from our results. For our static Monte Carlo ensemble the number of string loops *increases* as we approach the critical string density from above (see Table 1).

In the low density phase, we were able to construct thermodynamic measures of the loop statistics from their average length and the length fluctuations, related to as energy and specific heat respectively. The critical exponent of the divergence of the susceptibility with temperature was -2.43. The asymmetry parameter η at the phase transition was 1.24 ± 0.01 , which corresponds to a vacuum expectation value $\langle\phi\rangle = 0.12$, and a string density of 0.426 strings per tetrahedron, i.e. $\langle length/volume \rangle = 2.56/a^2$, where a is the edge length of the cubes of the body-centered cubic lattice, spanned by the vertices of the tetrahedra, so that the correlation length lies between $\sqrt{3}/2 a$ and a . This value is considerably higher than the one computed for differently constructed strings on a cubic lattice in ref. [15] ($\rho_c \approx 0.88/a^2$ there), which might again partly be caused by the fact that we allow smaller loops to form than there are on a cubic lattice. Nevertheless, one

does not expect the density to be universal for different ways of constructing strings (i.e. for discretizations of different symmetry groups), but further work on RP^2 is in progress and will shed more light on this question. Also the question whether the critical exponents exhibit universality with respect to the different ways of discretizing the vacuum manifold and space, or even some universality with respect to different vacuum manifolds, remains open.

We conclude with some discussion of the connections between our work and other problems in statistical physics. It is probably clear that what we are considering is a kind of percolation problem [29]. In fact, our field of three phases is essentially the spin of a 3-state Potts model, and the appearance of infinite string at the Hagedorn transition can only happen when the disfavoured spins ($k = 1, 2$) percolate. However, in conventional percolation theory one is generally interested in the domain size distribution $n(s)$ and its moments: here, we are picking out the junction between 3 domains for study. Nevertheless, there are close parallels. For example, the mean cluster size is equivalent to $\langle l^2 \rangle$, and the Fisher exponent τ for the cluster size distribution, defined by $n(s) \propto s^{-\tau}$ at criticality, is equivalent to our exponent b . The strength of the infinite cluster is related to the length in infinite string, ρ_∞ . Indeed, if we define a new exponent β from the non-analytic behaviour of ρ_∞ near the transition, such that

$$\rho_\infty \propto |\eta - \eta_c|^\beta, \quad (13)$$

these analogies suggest the following scaling relation:

$$\beta = (b - 2)/\sigma. \quad (14)$$

Another scaling relation should also hold:

$$\gamma = (3 - b)/\sigma. \quad (15)$$

Unfortunately, our statistics were not good enough to be able to extract β very satisfactorily. From Table 1 it can be seen that ρ_∞ is still non-zero for densities below the transition. This is an effect of the length cut-off – if we could go to arbitrarily long string, ρ_∞ would presumably vanish at η_c . The scaling relation (14) is also rather sensitive to values of b near 2. The form of ρ_∞/ρ , plotted in Figure 8, is nevertheless indicative of a second order transition.

The second of the above scaling relations is easier to check: with $b(\eta_c) = 2.11$ and $\sigma \simeq 0.397$, we should have $\gamma = 2.24$. Our measurements give $\gamma = 2.36$. We do not have sufficient confidence in our statistics to be able to say whether this is a significant deviation, but it is encouraging to find agreement to 5%.

Our simulations also model the initial conditions of condensed matter systems with a nonconserved order parameter after a rapid quench. Dynamical simulations have been performed on systems where the order parameter is a complex scalar field ϕ , both with $\langle\phi\rangle = 0$ (“critical”, corresponding to $\eta = 1$), and $\langle\phi\rangle \neq 0$ (“off-critical”, corresponding to $1 < \eta < 3$) [30]. It is found that the introduction of a bias in the initial expectation value of the order parameter results in the eventual departure from dynamical scaling, with the density of the string network going as

$$\rho(t) \sim t^{-1} \exp(-gt^{3/2}), \quad (16)$$

where g depends approximately quadratically on the initial bias $\langle\phi\rangle$. This is due to the network breaking up into isolated loops, with an exponentially suppressed size distribution. What is not clear is whether this is due to the initial conditions possessing no infinite string, or whether the infinite string somehow manages to chop itself up into an infinite number of loops. If it is the former, then that means that the percolation transition happens at very small bias, perhaps even $\langle\phi\rangle = 0$, for the departure from power-law scaling in $\rho(t)$ was observed from rather small biases, down to $\langle\phi\rangle = 0.001$. This brings us back to the question raised earlier this section: how does the discretisation affect the percolation transition? We plan to investigate this point further.

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Figure Captions

Figure 1: The tetrahedral subdivisions of a body-centered cubic lattice, which do fulfill the requirements of having almost regular tetrahedra (i.e. edges of almost equal length) and of creating a string network which is in a statistical sense isotropic, if the field values of the $U(1)$ manifold are assigned to the vertices of the lattice.

Figure 2: The dual lattice of the lattice in Figure 1. The edges of this lattice are the possible paths a string can take. Every vertex can be visited only once, since it sits in the center of a tetrahedron of the lattice in Figure one. This so-called tetrakaidekahedral lattice has the Brillouin zone of the body-centered cubic lattice as its elementary cell. The field values are assigned to the centers of these elementary cells. The near spherical shape of the elementary cell shows that this lattice is modelling the Kibble mechanism rather well, since the border of this cell represents half the distance to the event horizon.

Figure 3: The logarithmic plot of the square of the average distance to the origin of a point that is reached after having followed the string for a distance l . The slope of the linear fit to this curve is $2/D$, with D the fractal dimension of the strings.

Figure 4: Logarithmic plot of the number density of string mass in loops within a certain length-range. The slope of the linear fit to this gives one minus the exponent b in Eq. (3). We measure $b = 2.47 \pm 0.02$, corresponding - if scaling is assumed - to $D = 2.04 \pm 0.03$, which can be taken as a consistency check for the scaling assumption Eq. (1) rather than another convenient way of measuring the exponent D .

Figure 5: The logarithm of the loop susceptibility S , plotted against the asymmetry parameter η (solid circles). The line is a fit to the data, with $S = S_0(\eta - \eta_c)^{-\gamma}$, where $\gamma = 2.36$ and $\eta_c = 1.243$.

Figure 6: A plot of the parameter b in Eq. (4) vs. the density of strings. Vachaspati [15] stated that below the percolation threshold of the string density the measurements of b are consistent with a value of 2. We don't find evidence for such behaviour, but find

that b keeps decreasing monotonically with the string density.

Figure 7: The length scale on which the loop distribution is exponentially suppressed for long loops diverges as $\rho \rightarrow \rho_c -$. It is compared to the a fit of the form $c = c_0 |\eta - \eta_c|^{1/\sigma}$, with $\eta_c = 1.256$ and $\sigma = 0.397$. For $\eta > \eta_c$ there is no natural length scale to be measured, and the scaling distribution Eq. (3) is recovered.

Figure 8: The fraction of string which does not return to the origin (“infinite” string) ρ_∞/ρ , where ρ is given by (8), plotted against the asymmetry parameter. This fraction is in effect an order parameter for the transition.

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Figure 1:

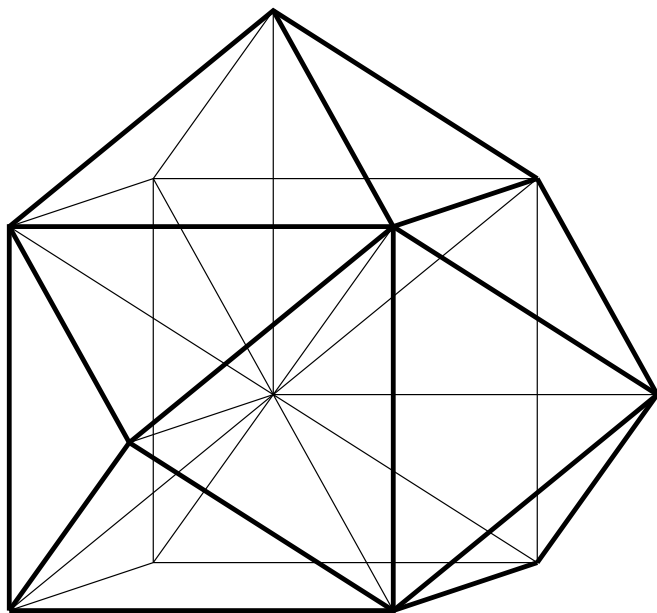
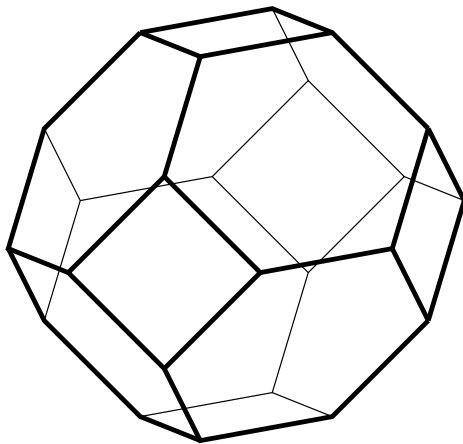


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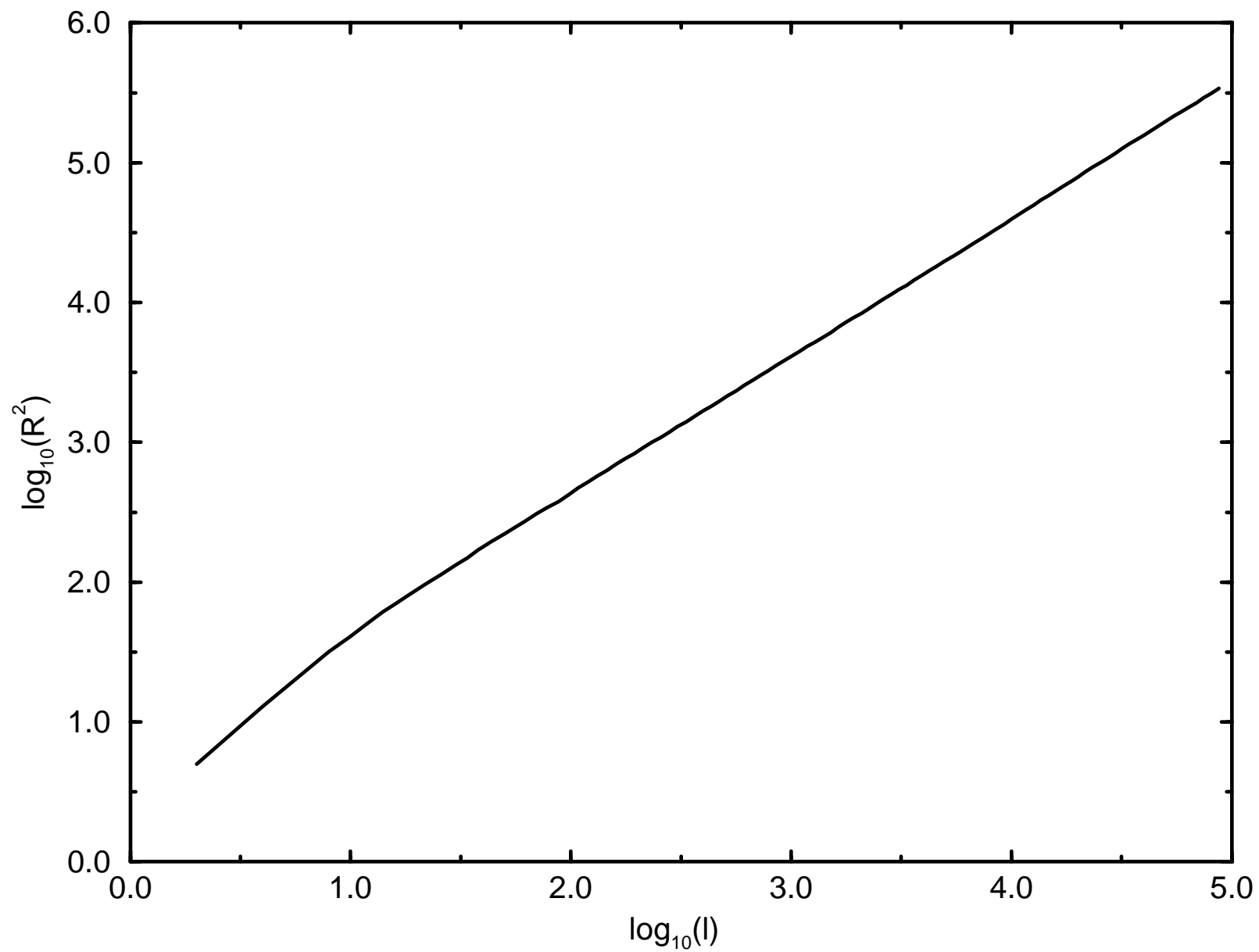
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Figure 3



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Figure 4

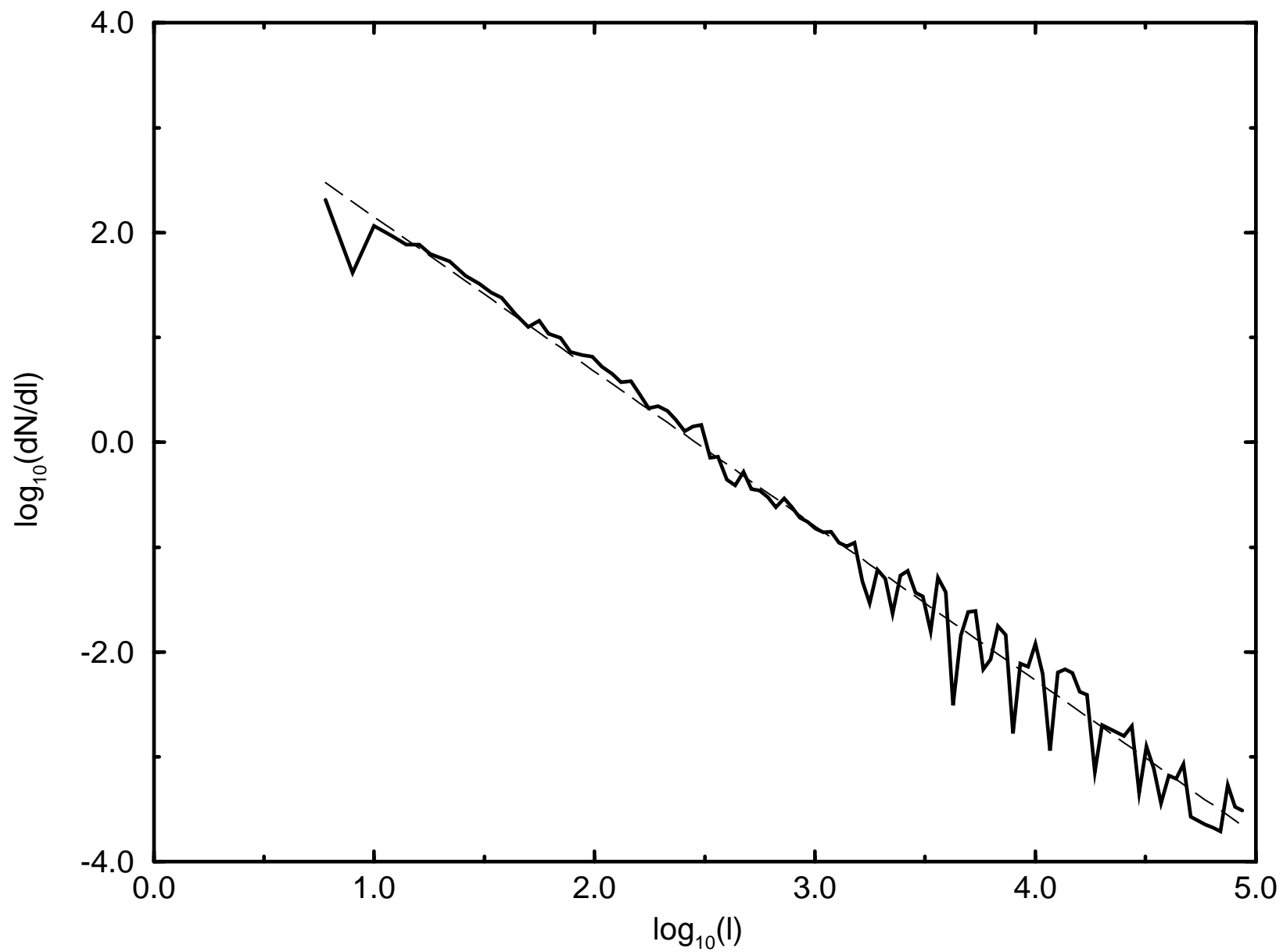


Figure 5

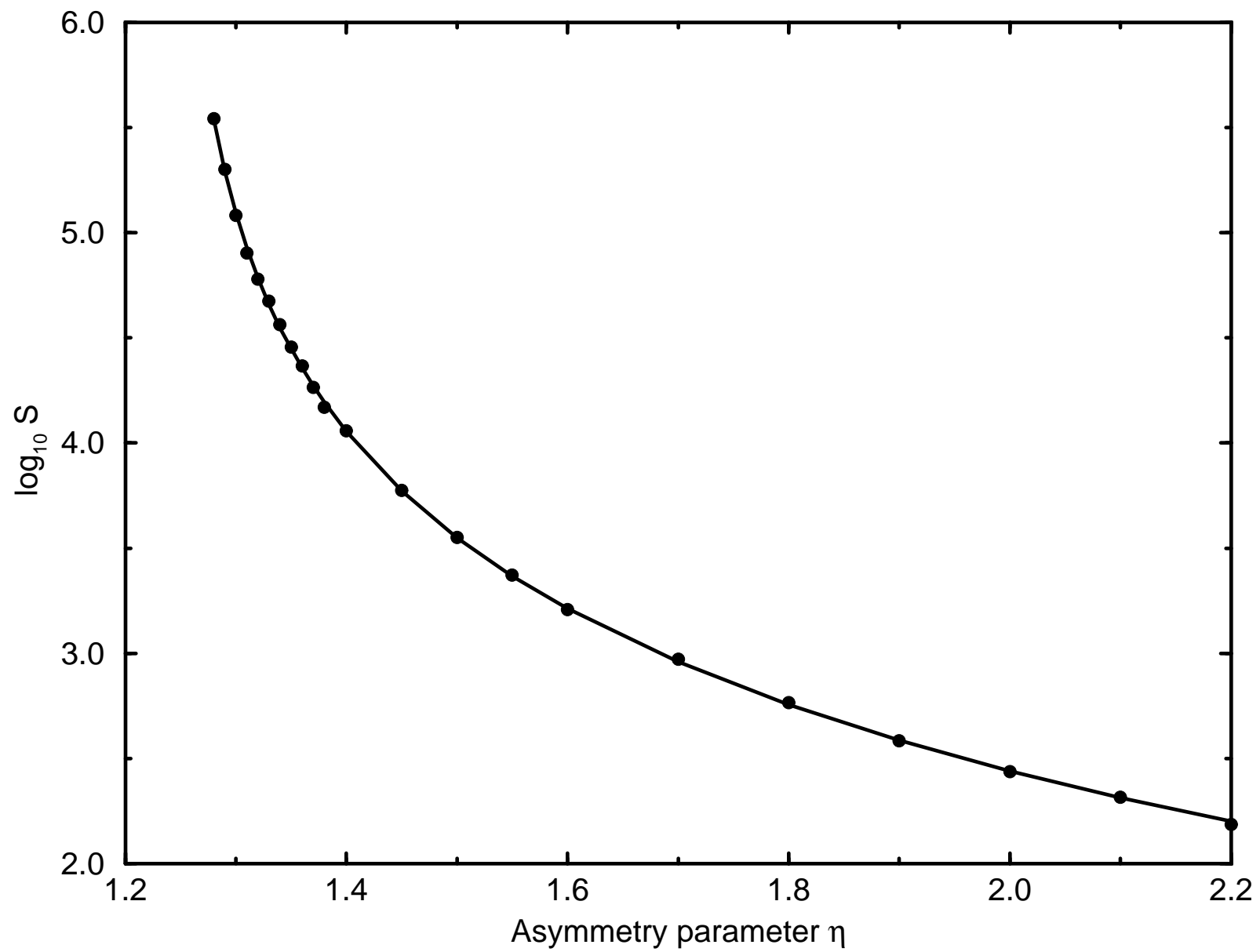


Figure 6

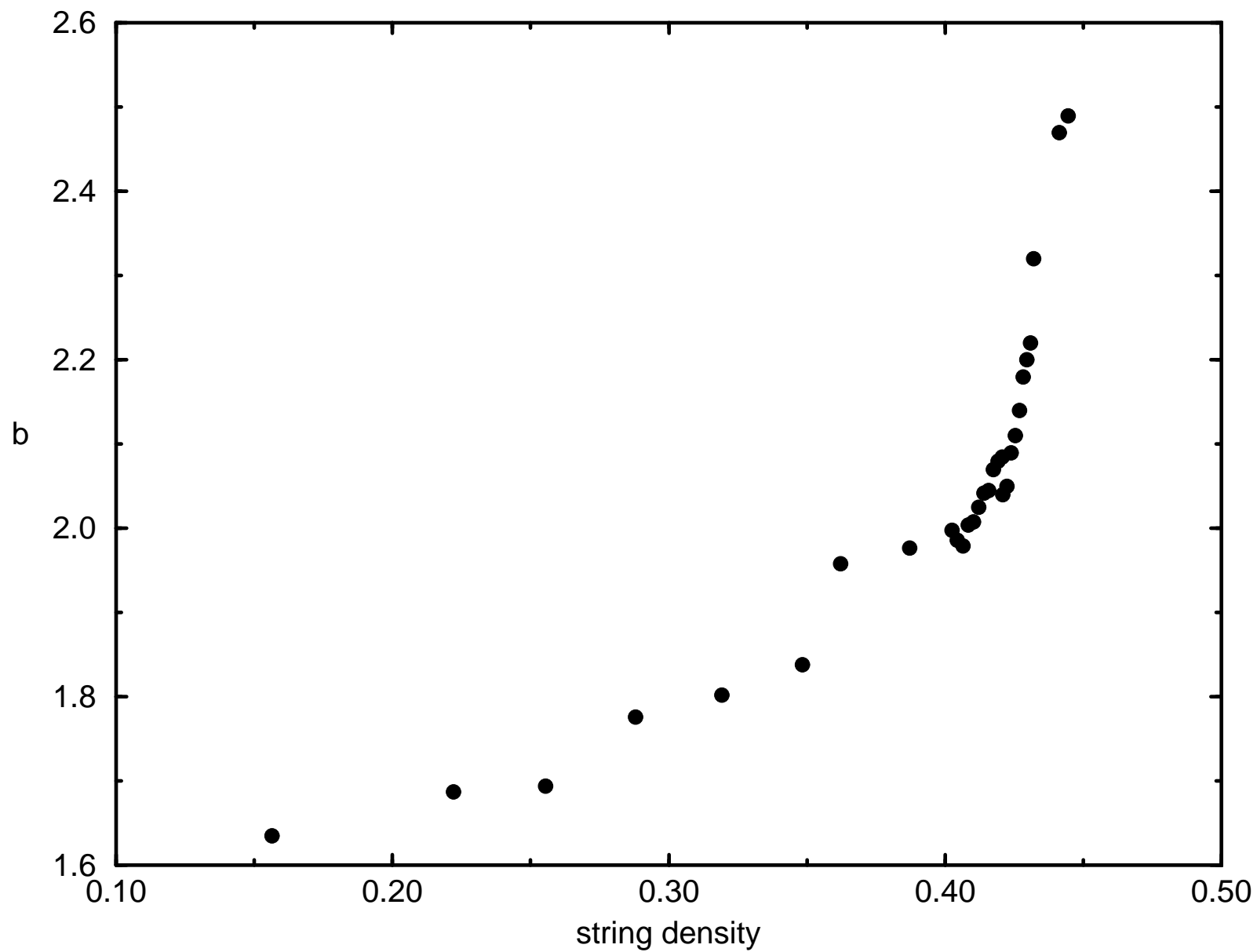


Figure 7

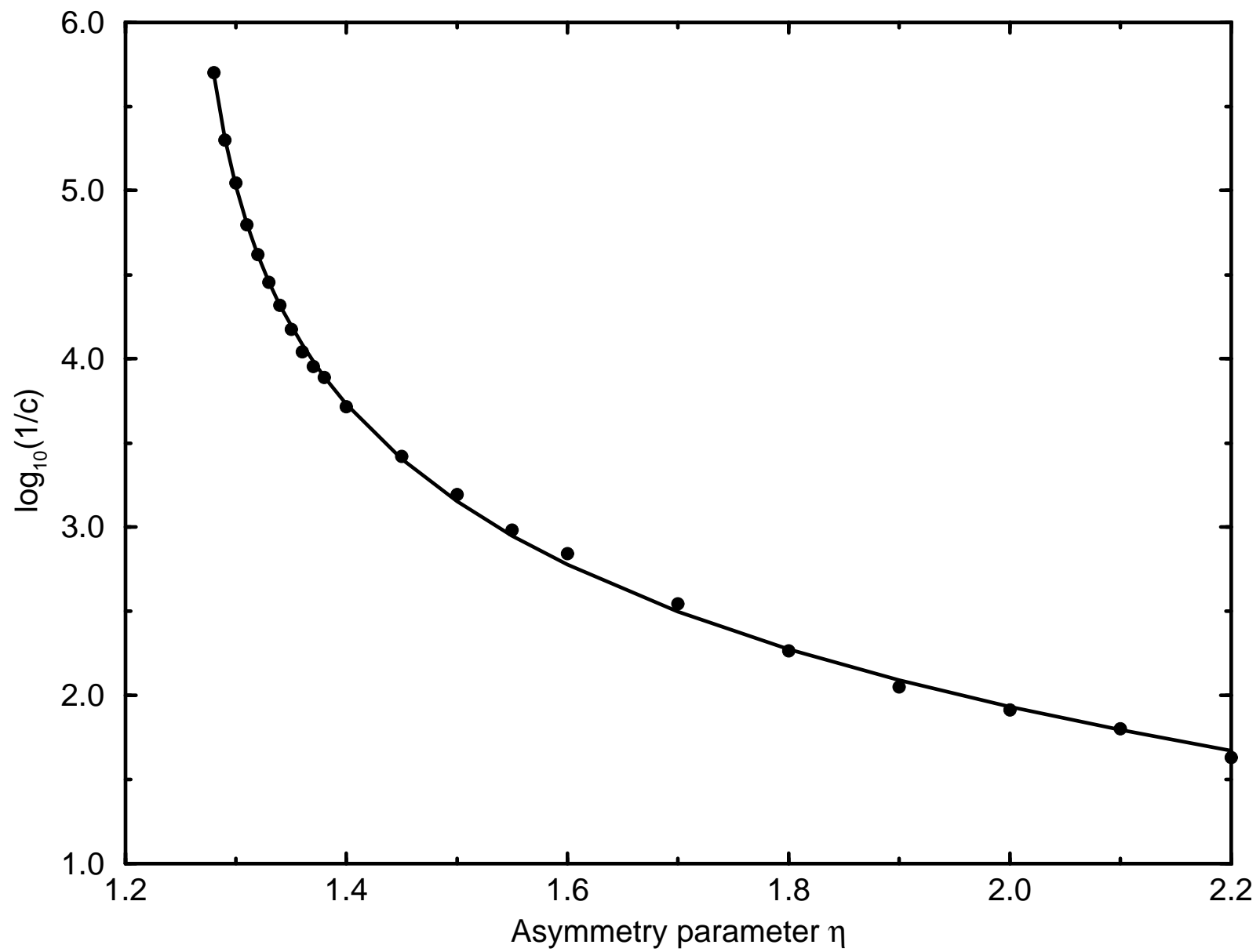


Figure 8

